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Notes

The Product of Affine Orthogonal Projections

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In the case of a finite number of subspaces in a given Hilbert space, by a theorem of J. von Neumann, the iteration of the product of projectors is always convergent. In a finite dimensional Hilbert space, this theorem has been generalized for affine subspaces. In this paper we construct an example which shows that this result does not hold in the infinite dimensional case. © 1991 Academic Press, Inc.

Let T_i^0 $(i = 1, 2, ..., m)$ denote the orthogonal projection operator onto a subspace H_i of a Hilbert space H. The following von Neumann theorem is well known (see [5, 8]).

THEOREM A. *For any x_0 in the Hilbert space H the alternating algorithm*

$$x_{k+1} = (T_m^0 T_{m-1}^0 \cdots T_1^0) x_k$$

converges as $k \to \infty$. Furthermore, the limit point is a fixed point of $T_m^0 \cdots T_1^0$.

Now let $V_i = H_i + f_i$ be affine subspaces of H, and let T_i be the affine orthogonal projection (metric projection) onto V_i, $i = 1, 2, ..., m$ We shall write "projector" instead of "orthogonal projection." In this respect Kosmol [6] has established

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THEOREM B. *If H is a finite dimensional Hilbert space, then the alternating algorithm*

$$x_0 \in H, \qquad x_{k+1} = (T_1 T_m T_{m-1} \cdots T_1) x_k \tag{1}$$

converges to a fixed point of $T_1 T_m \cdots T_1$, as $k \to \infty$.

Theorem B shows that for finite dimensional Hilbert spaces Theorem A is true for the product of affine projectors. Now, the question is whether Theorem B also holds in infinite dimensional Hilbert spaces. It is not difficult to see that if $\bigcap_{i=1}^m V_i \neq \varnothing$, von Neumann's theorem is also true for affine projectors (see [3]). Under this condition or something like this, a more general discussion can also be found in [2]. But what can we say if $\bigcap_{i=1}^m V_i = \varnothing$? We see that in this case $T = T_m \cdots T_1$ may not have any fixed points. To show this we first record the following result which can be obtained from a general theorem about nonexpansive mappings (see, e.g., [1, 7]). We write it now as

PROPOSITION 1. *The following three assertions are equivalent*:

(1) *T has a fixed point,*

(2) *the semiorbit $\{T^n(0) : n \in N\}$ is bounded in H, and*

(3) *there exists a nonempty bounded, closed, and convex subset K of H with $T(K) \subset K$.*

Now we are ready to mention our results. We have

PROPOSITION 2. *If $H_1 + H_2$ is closed, and $f_1, f_2 \in H$ then the product of the affine projectors $T_1 : H \to H_1 + f_1$ and $T_2 : H \to H_2 + f_2$ has* Fix $T_2 T_1 \neq \varnothing$.

Proof. Let us suppose that $f_1 \in H_1^\perp$ and $f_2 \in H_2^\perp$. In fact $H_i + f_i = H_i + f_i^\perp$, where $f_i^\perp \in H_i^\perp$, $f_i - f_i^\perp \in H_i$, and $i = 1, 2$. First, if $f_1 = 0$ we have

$$\text{Fix } T_1 T_2 = \{H_2 + H_1^\perp \cap H_2^\perp + f_2\} \cap H_1. \tag{2}$$

To prove this, let $x \in \text{Fix } T_1 T_2$. Obviously $x \in H_1$. On the other hand since $f_2 \in H_2^\perp$, we have for $y \in H$ and T_2^0 the projector onto H_2, $T_1 T_2 y = T_1 T_2^0 y + T_1 f_2$. Hence $T_1(T_2^0 x - x) = -T_1 f_2$. Putting $x = x_1 + x_2$, $x_1 \in H_2$, and $x_2 \in H_2^\perp$, we get for some $a \in H_1^\perp$, $x_2 = a + f_2$ and $x = x_1 + a + f_2$. Therefore

$$x \in \{H_2 + (H_1^\perp + f_2) \cap H_2^\perp\} \cap H_1 = \{H_2 + H_1^\perp \cap H_2^\perp + f_2\} \cap H_1.$$

Conversely, for $x \in \{H_2 + H_1^\perp \cap H_2^\perp + f_2\} \cap H_1$, $x = x_1 + x_2 + f_2$, $x_1 \in H_2$, $x_2 + f_2 \in H_2^\perp$, and $x_2 \in H_1^\perp$. Hence $x = T_1 T_2^0 x + T_1 f$ and $x \in \text{Fix } T_1 T_2$.

Now since $H_1 + H_2$ is closed, we obtain $f_2 = a_1 + a_2 + h$, $a_1 \in H_1$, $a_2 \in H_2$, and $h \in (H_1 + H_2)^\perp = H_1^\perp \cap H_2^\perp$. Put $x = -a_2 + (-h) + f_2$. Then $x \in (H_2 + H_1^\perp \cap H_2^\perp + f_2) \cap H_1$. That means Fix $T_1 T_2 \neq \emptyset$.

Denote by T_1^0, T_2^0 the projectors onto H_1, H_2, respectively, and $T_0 = T_2^0 T_1^0$. It follows from (2)

$$\text{Fix } T_1^0 T_2 \neq \emptyset, \qquad \text{Fix } T_2^0 T_1 \neq \emptyset.$$

But,

$$T^k x = T_0^k x + \sum_{j=0}^{k-1} T_0^j (T_2^0 f_1 + f_2). \tag{3}$$

Let $x_1 \in \text{Fix } T_1^0 T_2$. Then $T_1^0 f_2 = x_1 - T_1^0 T_2^0 x_1$ and

$$\left\| \sum_{j=1}^{N} T_0^j f_2 \right\| = \left\| \sum_{j=1}^{N} (T_2^0 T_1^0)^{j-1} (T_2^0 x_1 - T_2^0 T_1^0 x_1) \right\| \leqslant 2 \|x_1\|.$$

Let $x_2 \in \text{Fix } T_2^0 T_1$; then $T_2^0 f_1 = x_2 - T_2^0 T_1^0 x_2$ and

$$\left\| \sum_{j=1}^{N} T_0^j (T_2^0 f_1) \right\| = \left\| \sum_{j=1}^{N} (T_0^j x_2 - T_0^{j+1} x_2) \right\| \leqslant 2 \|x_2\|.$$

Hence

$$\sup_N \left\| \sum_{j=1}^{N} T_0^j (T_2^0 f_1 + f_2) \right\| \leqslant 2(\|x_1\| + \|x_2\|).$$

Thus, it follows from Proposition 1 and (3) that Fix $T \neq \emptyset$ or Fix $T_2 T_1 \neq \emptyset$. ∎

Similar to [9], let $\cos\theta = \sup |\langle x, y \rangle|$, for x and y in $H_1 \cap (H_1 \cap H_2)^\perp$ and $H_2 \cap (H_1 \cap H_2)^\perp$ with $\|x\| = \|y\| = 1$, respectively.

Proposition 3. *If* $\cos\theta < 1$, *then* Fix $T_2 T_1 \neq \emptyset$.

Proof. By [9], when $\cos\theta < 1$, there exists a constant $0 < C < 1$ such that

$$\|T_0^k x\| \leqslant C^k \|x\|$$

for $x \in (H_1 \cap H_2)^\perp$. Using (3) we get

$$\sup_k \|T^k(0)\| < \infty.$$

Hence Fix $T_1 T_2 \neq \emptyset$ by Proposition 1. ∎

The main result of this paper is the following

THEOREM. *There exist H, H_1, H_2, and $f \in H_2^\perp$ such that the product of the affine projectors $T_1: H \to H_1$ and $T_2: H \to H_2 + f$ has no fixed points, i.e.,*

$$\text{Fix } T_2 T_1 = \varnothing.$$

Proof. As in [4], let $H = l_2 \times l_2$, $H_1 = \{(y, Ay), y \in l_2\}$, and $H_2 = \{(y, 0), y \in l_2\}$ with $Ay = \{a_n y_n\}$, where $0 < a_n \leqslant 1$, $a_n \to 0$ as $n \to \infty$. Then H_1 and H_2 are closed subspaces. For some $\{a_n\}$, $H_1 + H_2$ is not closed. Let T_1 be the projector onto H_1. It is easy to see that, for $x = \{x_n\} \in l_2$,

$$T_1(0, x) = \left(\left\{ \frac{a_n x_n}{1 + a_n^2} \right\}, \left\{ \frac{a_n^2 x_n}{1 + a_n^2} \right\} \right) \tag{4}$$

and

$$T_1(x, 0) = \left(\left\{ \frac{x_n}{1 + a_n^2} \right\}, \left\{ \frac{x_n a_n}{1 + a_n^2} \right\} \right). \tag{5}$$

Put $f = (0, y) \in H_2^\perp$ with $y = \{y_n\}$, T_2 as the affine projector onto $H_2 + f$, and T_2^0 as the corresponding projector. We have Fix $T_2 T_1 = \varnothing$ for some $f \in H_2^\perp$. In fact, if Fix $T_2 T_1 \neq \varnothing$, then there exists $x \in \text{Fix } T_2 T_1$ such that

$$x = T_2 T_1 x = T_2^0 T_1 x + f \equiv T_0 x + f$$

and for $k \in N$, by (3)

$$x = (T_2 T_1)^k x = T_0^k x + \sum_{j=0}^{k-1} T_0^j f.$$

$T_0^k x$ is convergent as $k \to \infty$ due to Theorem A, so $\sum_{j=0}^{\infty} T_0^j f$ is convergent. On the other hand, by (4) and (5),

$$T_2^0 T_1 f = \left(\left\{ \frac{a_n y_n}{1 + a_n^2} \right\}, \{0\} \right)$$

and

$$T_0^k f = \left(\left\{ \frac{a_n y_n}{(1 + a_n^2)^k} \right\}, \{0\} \right),$$

so

$$\left\{ y_n a_n \sum_{k=1}^{\infty} \frac{1}{(1 + a_n^2)^k} \right\} \in l_2.$$

But $\{y_n a_n^{-1}\} \notin l_2$ if $a_n = y_n = 1/n$. Therefore $\text{Fix}\, T_2 T_1 = \emptyset$ for $f = (0, \{1/n\})$. The proof is complete. ▮

For the construction of Fix T, we have

PROPOSITION 4. *Let $T = T_m T_{m-1} \cdots T_1$ be the product of affine projectors in Hilbert space with* Fix $T \neq \emptyset$. *Then*

$$\text{Fix}\, T = \left\{ x_0 + x \;\middle|\; x \in \bigcap_{i=1}^{m} H_i \right\},$$

where x_0 is a fixed point of T.

Proof. Suppose $x_1, x_2 \in \text{Fix}\, T$. We obtain

$$x_1 - x_2 = Tx_1 - Tx_2 = T_0 x_1 - T_0 x_2 = T_0(x_1 - x_2).$$

So $x_1 - x_2 \in \text{Fix}\, T_0$. By [6], $x_1 - x_2 \in \bigcap_{i=1}^{m} H_i$. Hence there is a $z \in \bigcap_{i=1}^{m} H_i$ such that $x_1 = x_2 + z$. If $x_0 \in \text{Fix}\, T$, $x \in \bigcap_{i=1}^{m} H_i$, then by the definition of T

$$T(x_0 + x) = T_0(x_0 + x) + T(0) = x + T_0(x_0) + T(0) = x + x_0,$$

i.e., $x_0 + x \in \text{Fix}\, T$. ▮

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